JOURNAL OF DIFFERENTIAL EQUATIONS **132**, 87–106 (1996)
ARTICLE NO. 0172

On the Existence of the Compact Global Attractor for Semilinear Reaction Diffusion Systems on $\mathbb{R}^N$

Sandro Merino*

Department of Mathematics, Heriot-Watt University, Riccarton Campus, Edinburgh EH14 4AS, United Kingdom

Received October 25, 1995; revised May 15, 1996

We show that a class of reaction diffusion systems on $\mathbb{R}^N$ generates an asymptoti-

ence of the compact minimal attractor for reaction diffusion systems on $\mathbb{R}^N$ that contain appropriate weight functions. We also state conditions, which guarantee that the attractor has finite Hausdorff-dimension.

1. INTRODUCTION

During the last few decades concepts from the theory of dynamical systems have been successfully applied to reaction diffusion systems defined on bounded domains of $\mathbb{R}^N$ (cf. [11], [13], [14], [18], [26], [27]). The applicability of dynamical systems theory is due to the fact that quite generally parabolic evolution problems on bounded domains lead to semi-flows with relatively compact (semi-) orbits. We prefer to speak of a semi-flow rather than of a semigroup to avoid confusion with the concept of an analytic C_0-semigroup generated by a linear operator. The relative compactness of the semiorbits implies that ω-limit sets of bounded semiorbits are not empty, and that dissipative systems possess a compact global attractor. For analogous equations on unbounded domains essential new difficulties arise. We illustrate this point through the following simple example: Consider the initial value problem for the following reaction diffusion equation

$$\begin{cases} \partial_t u - \triangle u = u - u^3 & \text{in} \quad \mathbb{R}^N \times (0, T) \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^N. \end{cases} \tag{1}$$

* E-mail: sandro@ma.hw.ac.uk.

0022-0396/96 $18.00

We will see that (1) generates a semiflow

$$\varphi\colon \mathbb{R}^+ \times BUC(\mathbb{R}^N) \to BUC(\mathbb{R}^N),$$

or in other words a (semi-) dynamical system with infinite dimensional phase space $BUC(\mathbb{R}^N)$. It is well known that in the scalar case $(N=1)$ equation (1) admits travelling wave solutions, which connect the spatially homogeneous equilibria $u \equiv 0$ and $u \equiv 1$. This means that there are semiorbits on the unit sphere of $BUC(\mathbb{R}^N)$, which have empty ω-limit set. Therefore (1) does not have a compact global attractor in $BUC(\mathbb{R}^N)$.

In this paper we will discuss a class of problems on $\mathbb{R}^N$, which contain weight functions. As a simple example consider the following modification of (1) obtained by introducing a bounded and Hölder continuous weight function m.

$$\begin{cases} \partial_t u - \Delta u = m(x)u - u^3 & \text{in} \quad \mathbb{R}^N \times (0, T), \\ u(\,\cdot\,, 0) = u_0 & \text{in} \quad \mathbb{R}^N. \end{cases} \tag{2}$$

We assume that the positive part m^+ of m is in $C_0(\mathbb{R}^N)$—the Banach space of continuous functions on $\mathbb{R}^N$ that vanish at infinity—and that the negative part m^- is large and well distributed in the following sense: For any open subset $G \subset \mathbb{R}^N$ that contains arbitrarily large balls

$$\int_G m^-(x)\,dx = \infty.$$

This condition was introduced by Arendt and Batty (cf. [2]) and was subsequently studied for the time periodic case in [6] and [7]. It has proved to be useful in applications to various nonlinear parabolic evolution problems on $\mathbb{R}^N$ (cf. [15], [16], [22]).

Under the above assumption on m we will prove that the orbits of (2) have non-empty ω-limit sets and that there exists a unique *compact* minimal attractor $\mathscr{A}$. We will also see that the attractor $\mathscr{A}$ is the maximal compact invariant set which attracts each bounded set in $BUC(\mathbb{R}^N)$, i.e., a global attractor in the sense of [11]. Moreover, $\mathscr{A}$ is contained in the closed subspace $C_0(\mathbb{R}^N)$ of $BUC(\mathbb{R}^N)$. In fact, $C_0(\mathbb{R}^N)$ is an attractive and invariant subspace for (2). In Theorem 3.5 we will apply the abstract results of Mañé [20] to obtain that the Hausdorff-dimension of $\mathscr{A}$ is finite.

We note that for fixed $t_0 \in (0, \infty)$ the map $\varphi(t_0, \,\cdot\,)$ induced by the semiflow associated with (2) is not completely continuous. However, we will show that (2) generates an *asymptotically compact* semiflow or briefly a semiflow of class $\mathscr{AK}$. This class is discussed by O. Ladyzhenskaya in the monograph [18]. It is a natural generalization of the class of compact semiflows. The semiflow will also turn out to be *asymptotically smooth* (cf. [11] Section 3.2.). The definitions of asymptotical smoothness and asymptotical compactness in [11] and [18], respectively, are similar in spirit and the criteria given in Lemma 3.2.3 in [11] and in Theorem 3.3 in [18] to verify the respective properties are the same. We remark that we will not use that criterion directly and that for technical reasons we have preferred to use the concept of asymptotical compactness in this paper.

Often an integral representation of the solution via the variation of constants formula is used to obtain a-priori compactness properties of the semiflow. Dissipativity assumptions then imply the existence of the global attractor. In our approach we will first assume the existence of an attractive bounded set. Then we will use this attractive set to show the asymptotic compactness (and asymptotic smoothness) of the semiflow. The proof of our main result Theorem 2.4 is based on the fact that the Hölder spaces $BUC^{\alpha}(\mathbb{R}^N)$, $\alpha \in [0, 1)$, are *vector lattices* (cf. [24]). This strategy leads to simple proofs and circumvents difficulties we encountered in trying to verify the asymptotic compactness of the semiflow by using the variation of constants formula.

Finally, we note that a different approach to the question of the existence of attractors for parabolic evolution equations in $\mathbb{R}^N$ is given by Babin and Vishik in [3]. The authors work in weighted Sobolev (Hilbert) spaces and construct the solutions of the evolution problem on the unbounded domain by using the solutions of the corresponding problem on large balls (with Dirichlet boundary conditions) and by a suitable passage to the limit. They also consider unbounded initial conditions. In our aproach we use recent generation results in spaces of bounded continuous functions (cf. [19]; see also [15]) which allow us to solve the evolution problem on unbounded domains directly. We also note that a semiflow of class $\mathscr{AK}$ on $L^2(\mathbb{R}^N)$ and the existence of a finite dimensional compact global attractor was obtained in [1]. The approach in [1] is however again a Hilbert space approach and therefore quite different from the techniques we use in our exposition. In particular, we do not need to impose growth restrictions on the nonlinearities as is the case in [1] and [3]. Results on attractors for degenerate parabolic equations on unbounded domains are obtained in [8] for the phase space $L^1(\mathbb{R}^N)$. The approach in [8] is based on the Crandall–Liggett Theorem.

2. A SEMIFLMOW OF CLASS $\mathscr{AK}$

We study the initial value problem for the following reaction diffusion system.

$$\begin{cases} \partial_t u_k + \mathscr{A}_k(x, \partial)\, u_k = f_k(x, u) & \text{in} \quad \mathbb{R}^N \times (0, T], \\ u(x, 0) = u_0(x) & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{3}$$

where $n, N \in \mathbb{N}$, $k = 1, ..., n$ and $u := (u_1, ..., u_n)$. The following assumptions are made throughout this paper for $k = 1, ..., n$.

$$\mathscr{A}_k(x, \partial) := -\sum_{i,j=1}^{N} a_{ij}^{(k)}(x)\, \partial_i \partial_j + \sum_{i=1}^{N} a_i^{(k)}(x, t) \partial_i,$$

$$\text{where } a_{ij}^{(k)},\ a_i^{(k)} \in BUC^{\mu}(\mathbb{R}^N) \text{ for some } \mu \in (0, 1)$$

$$\text{and } \sum_{i,j=1}^{N} a_{ij}^{(k)}(x)\, \xi_i \xi_j \geqslant c_0\, |\xi|^2,\ c_0 > 0,$$

$$\text{for } x \in \mathbb{R}^N \text{ and } \xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^N. \tag{A1}$$

$$f_k(\,\cdot\,, \xi) \in BUC^{\mu}(\mathbb{R}^N) \text{ uniformly for } \xi \text{ in bounded subsets of } \mathbb{R}^N. \tag{A2}$$

For $i \in \{1, ..., n\}$ the derivatives $\partial_{\xi_i} f_k$ exist and are uniformly Lipschitz continuous on sets of the form $\mathbb{R}^N \times B$, where B is a bounded subset of $\mathbb{R}^n$. (A3)

Set $X := BUC(\mathbb{R}^N)$ and $\mathbf{X} := X^n$. We shall consider initial values $u_0 \in \mathbf{X}$.

Moreover, let $C_0^{\alpha}(\mathbb{R}^N)$, $\alpha \in [0, 1)$, be the Banach space of α-Hölder continuous real-valued functions that vanish at infinity. We note that X and $C_0{}^{\alpha}(\mathbb{R}^N)$ are *vector lattices*. For $u, v \in X$ we denote the *lattice operations* by

$$u \wedge v \qquad \text{and} \qquad u \vee v,$$

where

$$(u \wedge v)(x) := \max\{u(x), v(x)\} \qquad \text{and} \qquad (u \vee v)(x) := \min\{u(x), v(x)\}.$$

By defining the above operations componentwise also $\mathbf{X}$ becomes a vector lattice and since no confusion seems likely, we keep the notation $\wedge$ and $\vee$. The space X is an ordered Banach space with positive cone X^+ given by the pointwise nonnegative functions in X. For u and v in X we write $u \geqslant v$ if $u - v \in X^+$. We define an *order interval* in X by

$$[u, v] := \{w \in X \mid v \geqslant w \geqslant u\}.$$

Again these concepts can be extended to $\mathbf{X}$ naturally by considering components.

For $k = 1, ..., n$ we denote the X-realization of $\mathscr{A}_k(\,\cdot\,, \partial)$ by A_k. In [15] or [19] it is proved that $-A_k$ is the generator of an analytic C_0-semigroup on X. Moreover, the interpolation spaces obtained by continuous interpolation between the domain $D(A_k)$ (equipped with its graph-norm) and X are shown to be little Hölder spaces.

Let $\mathbf{A}$ be the diagonal operator defined by $(A_1, ..., A_n)$. Then $-\mathbf{A}$ generates an analytic C_0-semigroup on $\mathbf{X}$. For $u \in \mathbf{X}$ we define the superposition operator G_f by

$$G_f(u)(x) := (f_1(x, u(x)), ..., f_n(x, u(x))).$$

It is shown in [5] that our assumptions (A2) and (A3) imply the local Lipschitz continuity of G_f, i.e.,

$$G_f \in C^{1-}(\mathbf{X}, \mathbf{X}). \tag{4}$$

We may now reformulate (3) as an abstract semilinear initial value problem in the Banach space $\mathbf{X}$:

$$\begin{cases} \partial_t u + \mathbf{A}u = G_f(u), & t \in (0, T], \\ u(0) = u_0 \in \mathbf{X}. \end{cases} \tag{5}$$

The following existence and regularity result for (5) may be found in [15] or [19].

THEOREM 2.1. *For each $u_0 \in \mathbf{X}$ there exists a unique maximal solution $u(\cdot, u_0) \in C(J(u_0), \mathbf{X}) \cap C^1(\dot{J}(u_0), \mathbf{X})$, where $\dot{J}(u_0) := J(u_0) \setminus \{0\}$ and $J(u_0)$ is the maximal interval of existence and has either the form $[0, T]$ or $[0, t^+(u_0))$. Here $t^+(u_0) \in [0, T]$ is the positive escape time of the solution $u(\cdot, u_0)$. If the solution satisfies the a-priori estimate*

$$\|u(t, u_0)\|_X \leqslant C,$$

for some $C > 0$ and $t \in J(u_0)$, then $J(u_0) = [0, T]$, i.e., $u(\cdot, u_0)$ is a global solution. Moreover, $u(x, t) := u(t, u_0)(x) = (u_1(x, t), ..., u_n(x, t))$ is a classical solution of (3), *i.e., for $k = 1, ..., n$*

$$u_k \in BUC(\mathbb{R}^N \times [0, T]) \cap BUC^{2+\mu, 1+(\mu/2)}(\mathbb{R}^N \times (\varepsilon, T])$$

for any $\varepsilon > 0$.

For the sake of easy reference we introduce some standard concepts. We say that (5) generates a *global semiflow* $(\varphi, \mathbf{W})$ on $\mathbf{W} \subset \mathbf{X}$

$$\varphi : \mathbb{R}_+ \times \mathbf{W} \to \mathbf{W}, \qquad \varphi(t, w) := u(t, w),$$

if the following conditions are satisfied:

(i) $\mathbf{W}$ is closed in $\mathbf{X}$ and for each $w \in \mathbf{W}$ the solution $u(t, w)$ of (5) is global.

(ii) If $w \in \mathbf{W}$, then $u(t, w) \in \mathbf{W}$ for $t > 0$.

We note that by our assumption (A3) a global semiflow generated by (5) is differentiable with respect to $w \in \mathbf{W}$. For details see e.g. [5] Sections 16 and 18.

Following [18], we set $\gamma_t^+(x) := \{\varphi(s, x) \mid s \geqslant t\}$. Then $\gamma^+(x) := \gamma_0^+(x)$ is the *positive semiorbit* of x and for $\mathbf{A} \subset \mathbf{W}$ we write $\gamma^+(\mathbf{A}) := \bigcup_{x \in \mathbf{A}} \gamma^+(x)$. The global semiflow is *bounded* if $\gamma^+(\mathbf{B})$ is bounded for $\mathbf{B}$ bounded. By $\omega(x) := \bigcap_{t \geqslant 0} \overline{\gamma_t^+(x)}$ we denote the ω-limit set of x. Let $\mathbf{A}$, $\mathbf{M}$ be subsets of $\mathbf{W}$. We say that $\mathbf{A}$ attracts $\mathbf{M}$, if for any $\varepsilon > 0$ there exists a $t_1(\varepsilon)$ such that $\varphi(t, \mathbf{M}) \subset \mathcal{O}_\varepsilon(\mathbf{A})$ for $t \geqslant t_1(\varepsilon)$. Here $\mathcal{O}_\varepsilon$ is the ε-neighbourhood of $\mathbf{A}$ in $\mathbf{W}$. If $\mathbf{A}$ attracts each point in $\mathbf{W}$, then $\mathbf{A}$ is called a *global attractor* for $(\varphi, \mathbf{W})$. We say that $\mathbf{A}$ is a *global B-attractor* for $(\varphi, \mathbf{W})$, if $\mathbf{A}$ attracts each bounded subset of $\mathbf{W}$. We call $\mathbf{A}$ *invariant* if $\varphi(t, \mathbf{A}) = \mathbf{A}$ for $t \in \mathbb{R}_+$. Finally, a global semiflow is *dissipative* (respectively, *B-dissipative*) if it has a bounded global attractor (respectively, a bounded global B-attractor). We will use the following concept taken from [18]:

DEFINITION 2.2. A global semiflow $(\varphi, \mathbf{W})$ belongs to the *class* $\mathcal{AK}$ if it has the following property: For every bounded set $\mathbf{B} \subset \mathbf{W}$ such that $\gamma^+(\mathbf{B})$ is bounded, each sequence $(\varphi(t_j, x_j))$, with $x_j \in \mathbf{B}$ and $t_j \nearrow \infty$ as $j \to \infty$, is precompact.

The importance of the class $\mathcal{AK}$ is made clear by the following result from [18].

PROPOSITION 2.3. *Assume that* (5) *generates a bounded and dissipative global semiflow* $(\varphi, \mathbf{W})$ *of class* $\mathcal{AK}$. *Then* ω*-limit sets are non-empty, invariant and compact, and there exists a* (*unique*) *non-empty minimal* B*-attractor* $\mathcal{A}$. *Moreover,* $\mathcal{A}$ *is compact and invariant. If* $\mathbf{W}$ *is connected, then* $\mathcal{A}$ *and* ω*-limit sets are also connected.*

We note that in the above proposition the minimal B-attractor $\mathcal{A}$ is the maximal invariant bounded set in $\mathbf{W}$, which attracts each bounded set in $\mathbf{W}$. Hence $\mathcal{A}$ is a global attractor in the sense of [11].

We are now ready to give a criterion for testing whether a global semiflow generated by (5) is of class $\mathcal{AK}$.

THEOREM 2.4. *Assume that* (5) *generates a bounded global semiflow* $(\varphi, \mathbf{W})$ *on* $\mathbf{W} \subset \mathbf{X}$. *Let* $\mathbf{V} = [\underline{w}, \bar{w}]$ *be a non-empty order interval in* $\mathbf{W}$. *Furthermore, suppose that* $\underline{w}$, $\bar{w}$ *are elements of* $C_0^\alpha(\mathbb{R}^N)^n$ *for some* $\alpha \in (0, 1)$. *Then if* $\mathbf{V}$ *is B-attractive,* $(\varphi, \mathbf{W})$ *is a B-dissipative global semiflow of class* $\mathcal{AK}$. *In particular, the assertions of Proposition* 2.3 *hold and the attractor* $\mathcal{A}$ *is contained in* $\mathbf{W} \cap C_0(\mathbb{R}^N)^n$.

We remark that under the assumptions of this theorem the semiflow $(\varphi, \mathbf{W})$ is also asymptotically smooth as defined in [11] (Section 3.2.).

Proof. Let $\mathbf{B}$ be a bounded subset of $\mathbf{W}$ and $(t_k) \subset (0, \infty)$ such that $t_k \nearrow \infty$ as $k \to \infty$. Let (b_k) be a sequence in $\mathbf{B}$. We have to prove that the sequence $(x_k) := (\varphi(t_k, b_k))$ has a convergent subsequence. For $v \in \mathbf{W}$ and $t \in \mathbb{R}_+$ we define the part of φ in $\mathbf{V}$ by

$$\kappa(t, v) := (\varphi(t, v) \wedge \underline{w}) \vee \bar{w}.$$

The part of φ outside of $\mathbf{V}$ is then given by

$$\psi(t, v) := \varphi(t, v) - \kappa(t, v).$$

Hence

$$\psi \colon \mathbb{R}_+ \times \mathbf{W} \to \mathbf{X},$$

$$\kappa \colon \mathbb{R}_+ \times \mathbf{W} \to \mathbf{V},$$

and

$$\varphi = \psi + \kappa.$$

We first show that the sequence $(z_k) := (\kappa(t_k, b_k))$ in $\mathbf{V}$ has a convergent subsequence: We proceed similarly as in Lemma 3.2 of [17]. Since, by assumption, $\underline{w}$ and $\bar{w}$ are in $C_0^\alpha(\mathbb{R}^N)^n$ and since $(\varphi(t_k, b_k)) \subset BUC^2(\mathbb{R}^N)^n$ by the regularity result in Theorem 2.1, we can use the fact that the Hölder spaces $BUC^\mu(\mathbb{R}^N)$, $\mu \in [0, 1)$, are vector lattices to infer that

$$(z_k) \subset \mathbf{V} \cap C_0^\alpha(\mathbb{R}^N)^n.$$

In other words, the surgery that was performed to define κ preserves the Hölder regularity. Now using the fact that $C^\alpha(\bar{\Omega})$ is compactly embedded in $C(\bar{\Omega})$ for any bounded domain $\Omega \subset \mathbb{R}^N$, and by extracting a diagonal sequence, we see that there exists a subsequence (z_{k_j}) of (z_k) that converges uniformly on compact subsets of $\mathbb{R}^N$. Since $\underline{w}$ and $\bar{w}$ vanish at infinity it is easily verified that $z_{k_j} \to z$ in $\mathbf{X}$, i.e., uniformly on $\mathbb{R}^N$, for some $z \in \mathbf{V}$.

Next we consider the sequence $(y_k) := (\psi(t_k, b_k))$ in $\mathbf{X}$. Since, by assumption, $\mathbf{V}$ is a global B-attractor of $(\varphi, \mathbf{W})$, for any $\varepsilon > 0$ there exists a $t_1(\varepsilon) > 0$ such that $\varphi(t, \mathbf{B}) \subset \mathcal{O}_\varepsilon(\mathbf{V})$ for $t \geqslant t_1(\varepsilon)$. Thus for $b \in \mathbf{B}$ and $t \geqslant t_1(\varepsilon)$

$$\|\psi(t, b)\|_{\mathbf{X}} = \|\varphi(t, b) - (\varphi(t, b) \wedge \underline{w}) \vee \bar{w}\|_{\mathbf{X}} \leqslant \varepsilon.$$

I.e., $\psi(t, b) \to 0$ in $\mathbf{X}$ as $t \to \infty$, uniformly for $b \in \mathbf{B}$. Thus $y_k \to 0$ in $\mathbf{X}$ as $k \to \infty$. Finally, we consider the sequence (x_k) in $\mathbf{W}$. Note that for $k \in \mathbb{N}$

$$x_k = y_k + z_k.$$

Since we have shown that (z_k) has a convergent subsequence and since (y_k) converges to 0, we see that (x_k) has a convergent subsequence. This proves that $(\varphi, \mathbf{W})$ is a global semiflow of class $\mathscr{A}\mathscr{K}$. ∎

Remark 2.5. The results of this section remain valid if we choose the underlying space as $X := C_0(\mathbb{R}^N)$. In that case we have to assume in addition that

$$f_k(x, 0, ..., 0) = 0 \quad \text{for} \quad k = 1, ..., n \qquad \text{and} \qquad x \in \mathbb{R}^N. \tag{A4}$$

This assumption implies that (4) remains valid in the $C_0(\mathbb{R}^N)$-setting. The generation result for the space $C_0(\mathbb{R}^N)$ goes back to H. B. Stewart (cf. [25]).

3. APPLICATION TO A SINGLE EQUATION

In this section we consider (3) in the case $n = 1$. In the following we omit the index k. We will describe a class of nonlinearities that lead to a global semiflow (φ, X^+) of class AK. An instrumental property of single reaction diffusion equations is that they satisfy the parabolic maximum principle. We will mostly use it in the form of a comparison principle: If $u_0, \ v_0 \in X$ such that $u_0 \leqslant v_0$, then $\varphi(t, u_0) \leqslant \varphi(t, v_0)$ for $t \geqslant 0$. I.e., a global semiflow generated by (3) is order preserving. For further details and the concepts of (strict) sub- and supersolutions we refer to [17] (Section 2.D).

To formulate our assumptions we introduce the following set of coefficients (cf. [2]):

$$\mathscr{E} := \left\{ m \in X^+ : \int_G m(x)\, dx = \infty \text{ for any open } G \subset \mathbb{R}^N \right.$$
$$\left. \text{that contains arbitrarily large balls} \right\}.$$

Here we note that in the above definition of $\mathscr{E}$ the centers of the balls may be chosen at arbitrary points in the set G. In particular, any half space in $\mathbb{R}^N$ of the form $E_i(c) := \{x \in \mathbb{R}^N : x_i > c\}$ with arbitrary $i \in \{1,..., N\}$ and $c \in \mathbb{R}$ is a set that contains arbitrarily large balls. If a function $m \in \mathscr{E}$ models an absorption coefficient, then we may say that $m \in \mathscr{E}$ means that the absorption is large and well distributed over $\mathbb{R}^N$ since, in particular,

$$\int_{E_i(c)} m(x)\, dx = \infty$$

for each $i \in \{1, ..., N\}$ and $c \in \mathbb{R}$.

The following additional properties of $f(x, u)$ will be needed to formulate our result.

$$\text{There exists a constant } C_0 \geqslant 0 \text{ such that } f(x, c) \leqslant 0 \text{ but } f(\cdot, c) \not\equiv 0 \text{ for } c \geqslant C_0 \text{ and } x \in \mathbb{R}^N. \tag{A5}$$

$$f \text{ is of the form } f(x, \xi) = g(x, \xi)\xi. \text{ Moreover, for each } u \in X^+ \text{ the positive part } g^+(\cdot, u(\cdot)) \text{ of } g(\cdot, u(\cdot) \text{ is in } C_0(\mathbb{R}^N), \text{ whereas the negative part } g^-(\cdot, u(\cdot)) \text{ is in } \mathscr{E}. \tag{A6}$$

Assumption (A5) implies that large constants are strict supersolutions for (3) and assumption (A6) guarantees that our problem has the trivial solution $u \equiv 0$.

We will obtain the following result as an application of Theorem 2.4.

THEOREM 3.1. *Assume that f satisfies* (A5) *and* (A6). *Then* (3) *generates a bounded, B-dissipative global semiflow* (φ, X^+) *of class* $\mathscr{AK}$. *In particular, the assertions of Proposition* 2.3 *hold for* (φ, X^+), *and the compact minimal global B-attractor* $\mathscr{A}$ *is contained in* $X^+ \cap C_0(\mathbb{R}^N)$.

For the proof of this Theorem we will need the following Lemma.

LEMMA 3.2. *Assume that f satisfies* (A6). *Then any steady state* $w \in X^+$ *of* (3) *is in* $C_0(\mathbb{R}^N)$.

Proof. A proof of this Lemma may be found in [7] or [16] in the more general time-periodic situation. For the sake of completeness we include that proof here.

Let $w \in X^+$ be a steady state of (3). Set $b^+(x) := g^+(x, w(x))$ and $b^-(x) := g^-(x, w(x))$. By our assumption on g^- in (A6) we can apply the results in [2] (see also [7], [15]) to obtain that the X-realization A^- of

$$-\mathscr{A}(x, \partial) - b^-$$

generates an exponentially stable semigroup $(e^{tA^-})_{t \geqslant 0}$. In particular, the spectral radius $r(e^{tA^-})$ is smaller than 1 for any $t > 0$. For a fixed $T > 0$ we can use the variation of constants formula to obtain the following representation for w:

$$w = e^{TA^-}w + \int_0^T e^{(T-\tau)A^-} b^+ w \, d\tau,$$

or

$$w = (1 - e^{TA^-})^{-1} \int_0^T e^{(T-\tau)A^-} b^+ w \, d\tau.$$

Since by (A6) we have that $b^+w \in C_0(\mathbb{R}^N)$ and since the closed subspace $C_0(\mathbb{R}^N)$ of X is invariant under the semigroup $(e^{tA^-})_{t\geqslant 0}$, we obtain that $w \in C_0(\mathbb{R}^N)$. ∎

Proof of Theorem 3.1. The assumptions (A5) and (A6) imply that (3) has the trivial solution $u \equiv 0$ and that solutions of (3) to initial values $u_0 \in X^+$ are bounded and nonnegative by the parabolic maximum principle. Thus, by Theorem 2.1 the equation (3) generates a global semiflow (φ, X^+). In order to apply Theorem 2.4 we will construct a B-attractive order interval of the form $[0, \bar{w}]$, with a suitable $\bar{w} \in X^+ \cap C_0^{\alpha}(\mathbb{R}^N)$.

Let C_0 be the constant in (A5). We first show that for $c \geqslant C_0$ each solution $u(t, c)$ of (3) to the constant initial value $c \in X^+$, converges to the same steady state $\bar{w} \geqslant 0$. In fact, since by (A5) each constant $c \geqslant C_0$ is a strict supersolution it is well-known (cf. [23] Theorem 2.5.1) that each solution $u(t, c)$ is monotone decreasing in t. By applying Theorem 3.4 in [17] we obtain

$$\|u(t, c) - \bar{w}(c)\|_X \to 0 \qquad \text{as} \quad t \to \infty$$

for some steady state $\bar{w}(c) \geqslant 0$. Moreover, we find the uniform a priori estimate for such steady states

$$0 \leqslant \bar{w}(c)(x) < C_0 \qquad \text{for} \quad x \in \mathbb{R}^N \qquad \text{and} \qquad c \geqslant C_0. \tag{6}$$

Note now that by Lemma 3.1

$$\bar{w}(c) \in C_0(\mathbb{R}^N) \qquad \text{for} \quad c \geqslant C_0.$$

This together with (6) and the convergence of each solution $u(t, c)$, $c \geqslant C_0$, implies that for each $c \geqslant C_0$ there exists a $T(c) \geqslant 0$ such that

$$u(t, c) \leqslant C_0 \qquad \text{for} \quad t \geqslant T(c).$$

By the maximum principle this implies that $\bar{w}(c) \leqslant \bar{w}(C_0)$ for $c \geqslant C_0$. On the other hand for $c \geqslant C_0$

$$u(t, c) \geqslant u(t, C_0) \qquad \text{for} \quad t \geqslant 0,$$

and therefore $\bar{w}(c) \geqslant \bar{w}(C_0)$ for $c \geqslant C_0$. We thus find that $\bar{w}(c) = \bar{w}(C_0)$ for $c \geqslant C_0$. We conclude that the order interval

$$V := [0, \bar{w}(C_0)] \subset X^+ \cap C_0(\mathbb{R}^N)$$

is a global B-attractor for (φ, X^+). Finally, since $\bar{w}(C_0)$ is a steady state it obviously has the Hölder regularity required by Theorem 2.4. ∎

We now apply Theorem 3.1 to a class of polynomial nonlinearities.

EXAMPLE 3.3. (a) We consider the following equation with polynomial nonlinearities

$$\partial_t u + \mathscr{A}(x, \partial) u = (a(x) + b_0(x)\, p(u) - u^r) u \qquad \text{in} \quad \mathbb{R}^N \times (0, \infty), \tag{7}$$

where $r \geqslant 1$ is an integer, p is a polynomial of order less or equal $r-1$ with real coefficients, and $a, b_0 \in BUC^{\mu}(\mathbb{R}^N)$. Furthermore we assume that b_0 has compact support, that $a^+ \in C_0(\mathbb{R}^N)$ and that $a^- \in \mathscr{E}$.

We verify that Theorem 3.1 applies to this class of equations: Since $f(x, c) = (a(x) + b_0(x)\, p(c) - c^r) c$ tends to $-\infty$ as $c \to \infty$, uniformly for $x \in \mathbb{R}^N$, it is clear that (A5) holds. To verify (A6), let $u \in X^+$ and note that for $x \in \mathbb{R}^N$

$$g^+(x, u(x)) = [a(x) + b_0(x)\, p(u(x)) - u(x)^r]^+ \leqslant [a(x) + b_0(x)\, p(u(x))]^+.$$

Since b_0 has compact support we obtain that $g^+(\cdot, u(\cdot)) \in C_0(\mathbb{R}^N)$. Analogously we find that

$$g^-(x, u(x)) = [a(x) + b_0(x)\, p(u(x)) - u(x)^r]^- \leqslant [a(x) + b_0(x)\, p(u(x))]^-.$$

Since $\int_G a^-(x)\, dx = \infty$ for any $G \subset \mathbb{R}^N$ that contains arbitrarily large balls, and since b_0 has compact support we obtain $\int_G g^-(x, u(x))\, dx = \infty$ for such $G \subset \mathbb{R}^N$.

(b) If in the above example r is an even number, then we obtain a global semiflow on X rather than on X^+. In fact, if r is even, then sufficiently negative constants are strict subsolutions. The assertions of Theorem 3.1 remain valid for the semiflow on X.

(c) Equation (7) was studied in [16] for the special case that $r = 1$ and $p \equiv 0$. In that case (7) describes diffusive logistic growth on $\mathbb{R}^N$.

Remark 3.4. Let (φ, X^+) be the global semiflow generated by Example (7) in the simplest case $r = 1$ and $p \equiv 0$. The linearization of (7) at a steady state $w \in X^+$ is given by

$$\partial_t u + \mathscr{A}(x, \partial)\, u - b(x)\, u = 0 \qquad \text{in} \quad \mathbb{R}^N \times (0, \infty), \tag{8}$$

where $b(x) := a(x) - 2\, w(x)$. Note that $b^+ \in C_0(\mathbb{R}^N)$ and $b^- \in \mathscr{E}$. The linear equation (8) generates a global semiflow (χ, X) and it is well-known that for $(t, u) \in \mathbb{R}^+ \times X$

$$\chi(t, u) = \partial_u \varphi(t, w) u.$$

By decomposing χ as suggested by the proof of Lemma 3.2 we will verify—consult the proof of the next theorem—that (χ, X) is α-condensing, as defined in [11] (p. 37). Here α denotes the Kuratowski measure of noncompactness. Unfortunately we are not able to decide whether this remains true for the nonlinear semiflow (φ, X^+) itself. In the next theorem we will see that the information on the linearized flow is however sufficient to show that the attractor $\mathscr{A}$ found in Theorem 3.1 has finite Hausdorff-dimension (cf. [11] Theorem 2.8.1).

THEOREM 3.5. *Assume that f satisfies* (A5) *and* A6) *so that the assertions of Theorem* 3.1 *hold. Furthermore suppose that for $w \in \mathscr{A}$ we can decompose $\partial_u f$ as*

$$\partial_u f(x, w(x)) = h(x, w(x)) - m(x),$$

with $m \in \mathscr{E}$ Hölder continuous and $h(\cdot, w(\cdot)) \in C_0(\mathbb{R}^N)$. Then the global attractor $\mathscr{A}$ has finite Hausdorff-dimension.

Proof. Let (φ, X^+) be the global semiflow generated by (3) and fix an arbitrary $w_0 \in \mathscr{A} \subset X^+ \cap C_0(\mathbb{R}^N)$. We will show that for some fixed $T > 0$ we can write

$$\partial_u \varphi(T, w_0) \in \mathscr{L}(X)$$

as the sum of a strict (linear) contraction and a compact operator. The finite Hausdorff-dimension of $\mathscr{A}$ then follows from the abstract result of Mañé in [20] as stated in [11] Theorem 2.8.2. For further information consult also [12] and [21]. To obtain such a decomposition observe that for $u_0 \in X$ the linearization $\partial_u \varphi(t, w_0) u_0$ solves the (non-autonomous) variational equation

$$\begin{cases} \partial_t u + \mathscr{A}(x, \partial) u = \partial_u f(x, \varphi(t, w_0)(x)) u & \text{in} \quad \mathbb{R}^N \times (0, \infty), \\ u(0) = u_0. \end{cases} \tag{9}$$

By applying our decomposition of $\partial_u f$ we can rewrite (9) as

$$\begin{cases} \partial_t u + \mathscr{A}(x, \partial)\, u + m(x) u = b(t, x) u & \text{in} \quad \mathbb{R}^N \times (0, \infty), \\ u(0) = u_0, \end{cases} \tag{10}$$

where $b(t)(x) = b(t, x) := h(x, \varphi(t, w_0)(x))$. We note that by (A3)

$$b(\cdot) \in C^{\nu}([0, T], C_0(\mathbb{R}^N))$$

for some $\nu \in (0, 1)$. By Theorem 5.1. in [15] our assumption $m \in \mathscr{E}$ implies that the X-realization A^- of $-\mathscr{A}(x, \partial) - m(x)$ generates an exponentially stable analytic C_0-semigroup $(e^{tA^-})_{t \geqslant 0}$. By the variation of constants formula we obtain the following representation of $\partial_u \varphi(T, w_0) u_0$

$$\partial_u \varphi(T, w_0) u_0 = L u_0 + K u_0,$$

where

$$L := e^{TA^-} \qquad \text{and} \qquad K := \int_0^T e^{(T-t)A^-} b(t)\, \partial_u \varphi(t, w_0)\, dt.$$

Since $(e^{tA^-})_{t \geqslant 0}$ is exponentially stable we have the estimate

$$\|e^{tA^-}\|_{\mathscr{L}(X)} \leqslant M e^{-\omega t},\ t > 0,$$

for some $M \geqslant 0$ and $\omega > 0$. Thus by possibly choosing T larger we obtain that

$$\|L\|_{\mathscr{L}(X)} < 1.$$

To obtain the compactness of K we apply the perturbation results in [6] Section 5: We will consider $b(t, x)$ on the righthand side of (10) as a perturbation of the homogeneous problem. First note that for each $t \in [0, T]$ the function $b(t) \in C_0(\mathbb{R}^N)$ induces a multiplication operator $B(t) \in \mathscr{L}(X)$ by

$$(B(t) v)(x) := b(t, x)\, v(x).$$

If we set $D(A^-)$ for the Banach space obtained by equipping the domain of A^- with its graph norm, then it is readily verified that for each $t \in [0, T]$

$$B(t) \in \mathscr{L}(D(A^-), X)$$

is compact. In fact, by approximating $b(t)(\,\cdot\,) \in C_0(\mathbb{R}^N)$ by a sequence of functions with compact supports and by using compact embedding theorems for Hölder spaces on bounded domains, we see that B(t) can be approximated in $\mathscr{L}(X)$ by a sequence of compact operators. Hence B(t) is compact. The compactness of K is obtained by applying Proposition 5.4. in [6].

Finally, we apply Theorem 2.8.2 in [11] to the differentiable map $S := \varphi(T, \,\cdot\,)$ and the compact set $\mathscr{A}$: Note that S can be defined on the open set

$$\operatorname{dom}(S) := \{u \in X : t^+(u) > T\} \subset X,$$

which contains X^+ and thus also $\mathscr{A}$. By the invariance of the attractor $S(\mathscr{A})=\mathscr{A}$. For $w_0\in\mathscr{A}$ we have just proved that $DS(w_0)=L+K$. Hence $\mathscr{A}$ has finite Hausdorff-dimension. ∎

In particular, the theorem above applies to our example (7). To see this just set $m:=a^-$.

4. APPLICATION TO SYSTEMS

In this section we apply the results of Section 2 to some reaction diffusion systems. We will not formulate general conditions on system (3), which imply the applicability of Theorem 2.4 but we prefer to illustrate the results on some specific examples.

A Class of 2-Species Systems

A class of 2-species reaction diffusion systems on $\mathbb{R}^N$ has been studied by the author in [22] in the more general time-periodic case. We will now verify that in the autonomous case the results of Section 3 can be used to show that the class of equations studied in [22] generates semiflows of class $\mathscr{A}\mathscr{K}$ on $\mathbf{X}^+$, which possess compact global attractors. For the readers convenience we state the assumptions that were made in [22]. We consider the autonomous system

$$\begin{cases}\partial_t u_1+\mathscr{A}_1(x,\partial)u_1=a(x)u_1-b(x)\,g_1(x,u_1)u_1-h_1(x,u_1,u_2)u_1\\ \partial_t u_2+\mathscr{A}_2(x,\partial)u_2=d(x)u_2-f(x)\,g_2(x,u_2)u_2+h_2(x,u_1)u_2\end{cases}$$
$$\text{in } \mathbb{R}^N\times(0,\infty). \tag{11}$$

Following Section 6 in [22] we make the following assumptions.

$$a,\ d\in BUC^{\mu}(\mathbb{R}^N);\ a^+,\ d^+\in C_0(\mathbb{R}^N) \text{ and } a^-\in\mathscr{E} \tag{A7}$$

There exist constants $R_0>0$ and $c_0>0$ such that $d(x)\leqslant -c_0$ for $|x|\geqslant R_0$. (A8)

b and f are nonnegative and bounded away from zero on the supports of a^+ and d^+, respectively. (A9)

$g_i\in BUC^{\mu,2}(\mathbb{R}^N\times[0,R])$ for each $R>0$. Furthermore g_i is nonnegative and $g_i(\cdot,0)\equiv 0$. Finally, $\partial_2 g_i>0$ on $\mathbb{R}^N\times\mathbb{R}^+$ and $\lim_{\xi\to\infty} g_i(x,\xi)=\infty$ uniformly for $x\in\mathbb{R}^N$. (A10)

$h_1\in BUC^{\mu,2}(\mathbb{R}^N\times[0,R]^2)$ and $h_2\in BUC^{\mu,2}(\mathbb{R}^N\times[0,R])$ for each $R>0$. (A11)

$h_1(x, \xi, \eta) \geqslant 0$ for $(x, \xi, \eta) \in \mathbb{R}^N \times (\mathbb{R}^+)^2$. For each $x \in \mathbb{R}^N$ the positive part $h_2^+(x, \xi)$ of $h_2(x, \xi)$ is monotone increasing in ξ. (A12)

Finally, $h_1(\cdot, \cdot, 0) = h_2(\cdot, 0) \equiv 0$.

The following theorem is a natural complement to the results obtained in [22].

THEOREM 4.1. *Assume conditions* (A7)–(A12). *Then the system* (11) *generates a global semiflow* $(\varphi, \mathbf{X}^+)$ *of class* $\mathscr{AK}$. *This semiflow possesses a connected compact minimal global B-attractor* $\mathscr{A} \subset \mathbf{X}^+ \cap C_0(\mathbb{R}^N)^2$. *The Hausdorff-dimension of* $\mathscr{A}$ *is finite.*

Proof. The globality of nonnegative solutions of (11) is shown in [22] (Theorem 6.1) for initial data in $C_0(\mathbb{R}^N)^2$. The proof remains unchanged for initial data in $\mathbf{X}^+$. To prove that the global semiflow $(\varphi, \mathbf{X}^+)$ is of class $\mathscr{AK}$ we will apply our criterion (Theorem 2.4) and Theorem 3.1. More precisely, we will resort to the B-attractive order interval constructed in the proof of Theorem 3.1. The following arguments are closely related to the proof of Proposition 9.1.b) in [22].

Let $(u_1^0, u_2^0) \in \mathbf{X}^+$ and denote the solution of (11) to the initial value (u_1^0, u_2^0) by $(u_1(t), u_2(t))$. Since h_1 is nonnegative u_1 satisfies the inequality

$$\partial_t u_1 + \mathscr{A}_1(x, \partial) u_1 \leqslant a(x) u_1 - b(x)\, g_1(x, u_1) u_1 \tag{12}$$

for $(x, t) \in \mathbb{R}^N \times (0, \infty)$. Let w_1 denote the (global) solution of the initial value problem

$$\begin{cases} \partial_t w + \mathscr{A}_1(x, \partial) w = a(x) w - b(x)\, g_1(x, w) w & \text{in} \quad \mathbb{R}^N \times (0, \infty), \\ w(0) = u_1(0) = u_1^0. \end{cases} \tag{13}$$

Then it is a consequence of (12) and the parabolic maximum principle that

$$u_1(t) \leqslant w_1(t) \quad \text{in } X \tag{14}$$

for $t \geqslant 0$. Our assumptions on a, b and g_1 imply that Theorem 3.1 is applicable to (13). In particular, there exists a B-attractive order interval for (13) of the form $[0, \bar{u}_1]_X$ with $\bar{u}_1 \in C_0^\alpha(\mathbb{R}^N)$, $\alpha \in (0, 1)$. Hence, given $\varepsilon > 0$ we find $t_1(\varepsilon) > 0$ such that

$$u_1(t) \leqslant w_1(t) \leqslant \bar{u}_1 + \varepsilon \qquad \text{in} \quad X$$

for $t \geqslant t_1(\varepsilon)$. Then Assumption A12 implies that

$$h_2(x, u_1(x, t)) \leqslant h_2^+(x, \bar{u}_1(x) + \varepsilon)$$

for $(x, t) \in \mathbb{R}^N \times (t_1(\varepsilon), \infty)$. Thus we obtain the following inequality for u_2

$$\partial_t u_2 + \mathscr{A}_2(x, \partial) u_2 \leqslant [d(x) + h_2^+(x, \bar{u}_1(x) + \varepsilon)] u_2 - f(x)\, g_2(x, u_2) u_2 \quad (15)$$

for $(x, t) \in \mathbb{R}^N \times (t_1(\varepsilon), \infty)$. By Assumption A8 on d and by choosing a sufficiently small $\varepsilon > 0$ we can apply Theorem 3.1 to the equation

$$\partial_t w + \mathscr{A}_2(x, \partial) w = [d(x) + h_2^+(x, \bar{u}_1(x) + \varepsilon)\,] w - f(x)\, g_2(x, w) w$$
$$\text{in} \quad \mathbb{R}^N \times (0, \infty). \quad (16)$$

As a consequence we find a B-attractive order interval $[0, \bar{u}_2]_X$ for (16) with $\bar{u}_2 \in C_0^\alpha(\mathbb{R}^N)$. If w_2 denotes the (global) solution of the initial value problem

$$\begin{cases} \partial_t w + \mathscr{A}_2(x, \partial) w = [\, d(x) + h_2^+(x, \bar{u}_1(x) + \varepsilon)\,] w - f(x)\, g_2(x, w) w \\ w(t_1(\varepsilon)) = u_2(t_1(\varepsilon)), \end{cases}$$
$$\text{in} \quad \mathbb{R}^N \times (0, \infty)$$

then by (15) the parabolic maximum principle yields

$$u_2(t) \leqslant w_2(t) \qquad \text{in} \quad X \quad (17)$$

for $t \geqslant t_1(\varepsilon)$. The estimates (14) and (17) for u_1 and u_2 imply that the order interval in $\mathbf{X}^+$

$$V := [0, \bar{u}_1]_X \times [0, \bar{u}_2]_X$$

is B-attractive for $(\varphi, \mathbf{X}^+)$. In fact the construction of the order interval V does not depend on the particular initial value $(u_1^0, u_2^0) \in \mathbf{X}^+$ and it is easily seen from the above constructions that V attracts bounded sets in $\mathbf{X}^+$. Hence we can apply Theorem 2.4 to obtain the existence of the minimal compact B-attractor $\mathscr{A}$ for (11).

To show that the Hausdorff-dimension of $\mathscr{A}$ is finite we note that the variational equation associated with (11) has the form

$$\partial_t \psi + \mathscr{A}(x, \partial) \psi = B(t) \psi \qquad \text{in} \quad \mathbb{R}^N \times (0, \infty),$$

where

$$\mathscr{A}(x, \partial) := \begin{pmatrix} \mathscr{A}_1(x, \partial) & 0 \\ 0 & \mathscr{A}_2(x, \partial) \end{pmatrix}.$$

It is a little tedious but straightforward to verify that $B(t)$ can be decomposed as

$$B(t) = B_c(t) - m^-,$$

where

$$m^- := \begin{pmatrix} a^- & 0 \\ 0 & d^- \end{pmatrix}$$

is independent of t and

$$B_c(t) := [b_{ij}(t)]_{1 \leqslant i, j \leqslant 2}$$

with

$$b_{ij} \in C^\nu([0, T], C_0(\mathbb{R}^N))$$

for some $\nu \in (0, 1)$ and $T > 0$. The **X**-realization A^- of $-[\mathscr{A}(x, \partial) + m^-]$ generates an exponentially stable analytic C_0-semigroup. As in the proof of Theorem 3.5 we regard $B_c(t)$ as a perturbation. The rest of the proof is now identical to the arguments given in the proof of Theorem 3.5. ∎

Finally, we remark that some examples of systems of the form (11) are given in Section 11 of [22].

Competition Systems

In order to illustrate that the techniques presented in this paper are not restricted to systems involving only two species we consider the following example

$$\partial_t u_k + \mathscr{A}_k(x, \partial) u_k = \left(a_{k0}(x) - \sum_{i=1}^{n} a_{ki}(x) u_i \right) u_k \qquad \text{in} \quad \mathbb{R}^N \times (0, \infty), \tag{18}$$

$k = 1, ..., n$. For $k = 1, ..., n$ we assume that $a_{k0}^+ \in C_0(\mathbb{R}^N)$ and that $a_{k0}^- \in \mathscr{E}$. Moreover, we require that $a_{ik} \geqslant 0$ for $i, k = 1, ..., n$ and that for $k = 1, ..., n$ the following relation between the supports of the growth- and damping coefficients is satisfied

$$a_{kk} \text{ is bounded away from zero on } \operatorname{supp}(a_{k0}^+). \tag{19}$$

Under these assumptions we can prove the following Theorem.

THEOREM 4.2. *The competition system* (18) *generates a global bounded B-dissipative semiflow* $(\varphi, \mathbf{X}^+)$ *of class* $\mathscr{A}\mathscr{K}$. *In particular*, (18) *has a connected, compact minimal global B-attractor* $\mathscr{A} \subset \mathbf{X}^+ \cap C_0(\mathbb{R}^N)^n$. *The Hausdorff-dimension of* $\mathscr{A}$ *is finite.*

Proof. Since by (19) large constants are supersolutions for the single equations of system (18) and since the system has the trivial solution, the parabolic maximum principle implies that solutions of (18) to initial values in $\mathbf{X}^+$ are nonnegative and global. In Example 3.3 we have observed that each logistic equation

$$\partial_t u + \mathscr{A}_k(x, \partial) u = a_{k0}(x) u - a_{kk}(x) u^2 \qquad \text{in} \quad \mathbb{R}^N \times (0, \infty), \tag{20}$$

has a global B-attractor $V_k := [0, \bar{u}_k]$ in X^+, where $\bar{u}_k$ is in $C_0^\alpha(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$. As a consequence of the assumption that $a_{ki} \geqslant 0$ for k, $i = 1, ..., n$ and the parabolic maximum principle we obtain that

$$\mathbf{V} := \prod_{k=1}^{n} V_k$$

is a global B-attractor for (18). We can now apply Theorem 2.4. Following the lines of the proof of Theorem 3.5 it is verified that the Hausdorff-dimension of $\mathscr{A}$ is finite. ∎

5. CONCLUDING REMARKS

A problem that has received much attention in the recent literature is the analysis of the structure of the global attractor of a scalar reaction diffusion equation on a bounded interval with Dirichlet or Neumann boundary conditions (cf. [4], [9], [10]). It appears to be an open question whether similar results are valid for weighted scalar equations on the whole real line.

In Section 2 we assumed that the nonlinearities in (3) only depend on u and not on ∇u. This restriction is essential for the applicability of the techniques that we used in the proof of Theorem 2.4. In fact, if f would depend on ∇u, then we would be forced to work in a space of differentiable functions (see e.g. [5] or [19]). However, such spaces are no vector lattices anymore and our proof of Theorem 2.4 therefore fails in that case. We remark that the gradient dependent case is not treated by Babin and Vishik (cf. [3]) either, whereas (under growth restrictions) it is allowed in [1].

Following the lines of Section 3.6 in [11], the ideas of this paper could also be applied to nonautonomous problems with periodic time dependence. For the existence result for the evolution system (periodic process) $U(t, s)$ we refer to [15].

ACKNOWLEDGMENTS

I thank B. Fiedler for inspiring discussions at the Equadiff 95 in Lisbon. I am also grateful to P. Poláčik for instructive comments during a visit at Heriot-Watt University. Finally, I thank Rachel Dunwell for some helpful remarks. This work was supported by the Heriot-Watt University and the European Reaction Diffusion Network in the framework of the Human Capital and Mobility program of the EU.

REFERENCES

1. F. Abergel, Existence and finite dimensionality of the global attractor for evolution equations on unbounded domains, *J. Differential Equations* **83** (1990), 85–108.
2. W. Arendt and C. J. K. Batty, Exponential stability of a diffusion equation with absorption, *Differential Integral Equations* **6** (1993), 1009–1024.
3. A. V. Babin and M. I. Vishik, Attractors of partial differential evolution equations in an unbounded domain, *Proc. Roy. Soc. Edinburgh Sect. A* **116** (1990), 221–243.
4. P. Brunovský and B. Fiedler, Connecting orbits in scalar reaction diffusion equations II: The complete solution, *J. Differential Equations* **81** (1989), 106–135.
5. D. Daners and P. Koch Medina, "Abstract Evolution Equations, Periodic Problems and Applications," Longman Scientific & Technical, Pitman Research Notes in Mathematics Series 279, Harlow, Essex, 1992.
6. D. Daners and P. Koch Medina, Superconvexity of the evolution operator and eigenvalue problems on $\mathbb{R}^N$, *Differential Integral Equations* **7** (1994), 235–255.
7. D. Daners and P. Koch Medina, Exponential stability, change of stability and eigenvalue problems for linear time periodic parabolic equations on $\mathbb{R}^N$, *Differential Integral Equations* **7** (1994), 1265–1284.
8. E. Feireisl, Asymptotic behaviour for degenerate parabolic equations on unbounded domains, *Nonlinear Anal. TMA* **26** (1996), 1535–1546.
9. B. Fiedler, Global attractors of one-dimensional parabolic equations: sixteen examples, *Tatra Mountain Math. Publ.* **4** (1994), 67–92.
10. B. Fiedler and J. Mallet-Parret, A Poincaré Bendixon theorem for scalar reaction diffusion equations, *Arch. Rational Mech. Anal.* **107** (1989), 325–345.
11. J. K. Hale, "Asymptotic Behaviour of Dissipative Systems," Amer. Math. Soc., Providence, RI, 1988.
12. J. K. Hale, L. T. Magalhães, and W. M. Oliva, "An Introduction to Infinite Dimensional Dynamical Systems-Geometric Theory," Series in Applied Mathematical Sciences, Vol. 47, Springer, Berlin, 1984.
13. D. Henry, "Geometric Theory of Semilinear Parabolic Equations," Lecture Notes in Mathematics, Springer, Berlin, 1981.
14. P. Hess, "Periodic-Parabolic Boundary Value Problems and Positivity," Longman Scientific & Technical, Pitman Research Notes in Mathematics Series 247, Harlow, Essex, 1991.
15. M. Hieber, P. Koch Medina, and S. Merino, Linear and semilinear parabolic equations on $BUC(\mathbb{R}^N)$, *Math. Nachr.* **179** (1996), 107–118.
16. M. Hieber, P. Koch Medina, and S. Merino, Diffusive logistic growth on $\mathbb{R}^N$, *Nonlinear Anal. TMA* **27**, No. 8 (1996), 879–894.
17. P. Koch Medina and G. Schätti, Longtime behaviour of reaction–diffusion equations on $\mathbb{R}^N$, *Nonlinear Anal. TMA* **25**, No. 8 (1995), 831–870.
18. O. Ladyzhenskaya, "Attractors for Semigroups and Evolution Equations," Lezioni Lincee, Cambridge Univ. Press, Cambridge, 1991.
19. A. Lunardi, "Analytic Semigroups and Optimal Regularity in Parabolic Problems," Birkhäuser, Basel, 1995.

20. R. Mañé, "On the Dimension of the Compact Invariant Sets of Certain Nonlinear Maps," Lecture Notes in Mathematics, Vol. 898, pp. 230–242, Springer, Berlin, 1981.
21. J. Mallet-Parret, Negatively invariant sets of compact maps and an extension of a theorem of Cartwright, *J. Differential Equations* **22** (1976), 331–348.
22. S. Merino, Positive periodic solutions for semilinear reaction diffusion systems on $\mathbb{R}^N$, *Adv. Differential Equations* **1** (1996), 579–609.
23. D. Sattinger, "Topics in Stability and Bifurcation Theory," Lecture Notes in Mathematics, Vol. 309, Springer, Berlin, 1973.
24. H. H. Schaefer, "Banach Lattices and Positive Operators," Springer, Berlin, 1974.
25. H. B. Stewart, Generation of analytic semigroups by strongly elliptic operators under general boundary conditions, *Trans. Amer. Math. Soc.* **259** (1980), 299–310.
26. R. Temam, "Infinite-Dimensional Systems in Mechanics and Physics," Springer, Berlin, 1988.
27. M. I. Vishik, "Asymptotic Behaviour of Solutions of Evolutionary Equations," Lezioni Lincee, Cambridge Univ. Press, Cambridge, 1992.